JOURNAL OF COMPUTATIONAL PHYSICS **135,** 217–219 (1997)
ARTICLE NO. CP975746

Introduction to "Contour Dynamics for the Euler Equations in Two Dimensions"

Zabusky, Hughes, and Roberts [32] introduced the method of "contour dynamics" to describe the motion of vorticity contours in an inviscid two-dimensional flow. When the distribution of vorticity is piecewise uniform, the contours coincide with vorticity jumps, and the governing equations may be derived without approximation from Euler's equations. Alternatively, one may view contour dynamics (hereafter CD) as an approximation to the dynamics of a flow having continuous vorticity. Such an approximation is much better than one might expect on first estimates [17] and has led to unforeseen practical uses of CD.

Contour dynamics grew out of an earlier model, called the "Water Bag Model," introduced a decade earlier by Berk and Roberts [2] in the context of the two-dimensional Vlasov equation in plasma physics (in fact, the idea goes even further back to a technical report of Dory in 1962 [34]—see the comments in [2]—and the name "contour dynamics" was coined by Harlow in 1971). Like Euler's equations in fluid dynamics, there is a materially conserved field (a distribution density) in a space spanned by particle position and momentum. However, the relationship between this field and the associated rates of change of particle position and momentum is significantly more complicated than the relationship between vorticity and velocity in a fluid. Zabusky *et al.* [32] capitalized on this fact and showed that, in CD, the velocity field $\mathbf{u}$ can be expressed succinctly in terms of contour integrals along the vorticity discontinuities $\Delta\omega_k$ with a weighting factor given by the Green function G of Laplace's operator:

$$\mathbf{u}(\mathbf{x}) = -\sum_k \Delta\omega_k \oint_{\mathscr{C}_k} G(\mathbf{x} - \mathbf{X}_k)\, d\mathbf{X}_k \tag{1}$$

where $\mathbf{X}_k$ is a point on contour $\mathscr{C}_k$ traversed in the right-handed sense, and $\Delta\omega_k$ is the inward jump in vorticity ω. For an unbounded fluid, $G(\mathbf{x} - \mathbf{X}_k) = (2\pi)^{-1} \log|\mathbf{x} - \mathbf{X}_k|$. Equation (1) expresses the fact that the contours alone determine the velocity field *everywhere.* The flow evolution, therefore, can be reduced to the motion of the contours, simple advection:

$$d\mathbf{X}_k/dt = \mathbf{u}(\mathbf{X}_k). \tag{2}$$

In fact, it turns out that these equations apply to a wide range of related systems [9] and new applications continue to be found. The essential ingredients are (1) a materially conserved quantity ω and (2) a linear "inversion" operator relationship between ω and the "velocity" field $\mathbf{u}$. The first ingredient permits one to represent ω as a piecewise-uniform function and to follow its evolution by simple advection, Eq. (2). The second ingredient allows one to reduce the calculation of $\mathbf{u}$ to contour integrals, Eq. (1) (although there are systems, like the Vlasov system, in which a generalized form of Eq. (1) is required; see [9]).

Examples of the diversity of CD include studies of ionospheric plasma clouds [22, 23], axially-stretching flows [25], axisymmetric flow without swirl [24, 28], helical flow without swirl [10], flow on the surface of a sphere [6], and multilayer quasi-geostrophic flow in a 3D, rotating, stratified fluid [11]. At present, in fact, it is in the arena of atmospheric and oceanic dynamics where CD and its extensions are most widely used.

The conceptual simplicity of CD has inspired numerous analytical studies, such as localized equations of the modified-KdV type for describing wave propagation on contours [14, 21]; semi-analytical studies, such as finding equilibrium contour configurations and assessing their stability (see [31, 9, 26, 27, 12]); approximate models of vortex behaviour, such as the moment model [19] and the elliptical model [16]; and wide-ranging attempts to prove, or disprove, the regularity of CD, culminating rigorously in favor of regularity ([3] and references therein).

Of course the interest in CD lies in its potential as a computational tool. For a number of problems, CD offers clear numerical and conceptual advantages: an effective reduction in the dimension of the system from two to one as well as a means for dealing with fields having sharp gradients. These advantages can lead to greatly reduced computational costs as well as greatly enhanced accuracy compared with conventional (pseudo-spectral or grid-point) numerical methods, particularly so for flows which can be represented by a small number of well-behaved contours.

These advantages were exploited early on in the algorithms developed by Berk and Roberts [2] and Zabusky *et al.* [32], and later by Deem and Zabusky [5] (published however a year earlier!), and by many others since [9; 1; 26, and references therein]. There are probably thousands

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of researchers that now use some form of CD or an extension thereof in their research, particularly in the atmospheric and oceanic sciences.

But a difficulty was also recognized early on—the tendency for contours to develop fine scale (foreseen, remarkably, by Kelvin in the last century; see [7] for the physical mechanism involved). This fine scale causes contour lengths to grow rapidly, and to maintain accuracy one has to continually add computational points. This slows down the simulation, since the computational cost in calculating the velocity at all points is proportional to the square of the number of points. If points are not added, contours begin to cross and the simulation accuracy degrades rapidly.

Berk and Roberts [2] introduced what they called "trimming" to slow the growth in contour complexity. This trimming reconnected, under certain circumstances, close contours enclosing the same ω. This permitted them to take their simulations to longer times, while incurring little loss in accuracy. Twenty years later, Dritschel [8] redeveloped this idea under the name "contour surgery" (CS), a robust, efficient method for limiting the complexity of contours (see also [9] and, more recently, [13]). Contour surgery ties together the way in which contours are topologically reconnected, below a prescribed scale δ, with the way in which points are distributed (and redistributed) along the contours. This is done to balance the accuracy of the various parts of the numerical algorithm, and thereby minimize computational expense for a prescribed accuracy. That one can perform accurate simulations with surgery is a consequence of the general insensitivity of $\mathbf{u}$ to fine scale ω.

An alternate way of controlling the development of fine-scale structure was introduced by Zabusky and Overman [33], but has not been fully explored. They introduced "tangential regularization," in which the contour curvature is diffusively damped. This limits the maximum curvature that can occur and controls contour stretching. However, this scheme does not permit contour reconnection, which may still occur, for instance, when two parts of the contour come closer than the viscous scale.

Contour dynamics has had one further extension recently, an extension which may prove highly practical, for example, in changing the way weather forecasting is now done. The use of CD/CS has been steadily growing in the atmospheric and oceanic scientific communities, to the point where it is now used more by these communities than by any other—certainly something which had not been foreseen early on. There are literally hundreds of published works using this algorithm, and there is extensive research in progress. A pivotal development was made simultaneously by Waugh and Plumb [29] and Norton [20], who converted the CS algorithm into a diagnostic tool for studying tracer advection in the atmosphere (to obtain new insight into the mechanisms leading to ozone depletion, see [18]). They replaced the computation of the velocity field via Eq. (1) by a given velocity field on a fixed grid, as one would have it from routine observational analyses or from a "global circulation model." They determined the (conservative) tracer evolution then by simple advection, Eq. (2). They called this technique "contour advection." Applications to the real stratosphere have far surpassed expectations: Contour advection revealed, for the first time, a proliferation of fine, sub-grid scale filamentary tracer structure, corroborated by high-resolution aircraft measurements (Waugh *et al.* [30]).

The success of this diagnostic tool motivated Dritschel and Ambaum [13] to reintroduce the dynamical calculation of the velocity field, not by Eq. (1) as in CD, but through the use of an underlying grid, as in the particle-in-cell algorithm for point vortices [4]. The contour advection studies have confirmed that the large-scale (or super-grid scale) velocity field dominates in advection: sub-grid scale fluctuations are practically negligible. Dritschel and Ambaum [13] further exploited the fact that fine-scale ω ("potential vorticity" in atmospheric or oceanic dynamics) is practically negligible for determining the velocity field, by virtue of the inversion operator. Dritschel and Ambaum [13] closed the circuit by introducing a novel, fast algorithm, linearly proportional to the number of points representing the contours, for converting the ω contours to gridded values. These gridded values are then used in a conventional way to calculate the gridded velocity field. The point is that, for all but the simplest flows, this can be done much more rapidly than via Eq. (1).

This hybrid algorithm, called the "Contour-Advective Semi-Lagrangian Algorithm," represents a computational breakthrough. It frees CD from the constraint that there must be a linear operator relationship between ω and $\mathbf{u}$. It marries the ideal elements of CD and conventional numerical methods. It permits one to study much more complex and much more realistic flows, e.g., atmospheric and oceanic flows having gravity waves and large-scale vorticity gradients, much more accurately than conventional algorithms for the same computational cost. Who would have thought that CD would have led to this?

ACKNOWLEDGMENT

DGD is supported by the UK Natural Environment Research Council.

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